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# Proof of universality of the Bessel kernel for chiral matrix models in the microscopic limit

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**Abstract**

We prove the universality of correlation functions of chiral complex matrix models in the microscopic limit ($N \to \infty,\ z \to 0,\ Nz =$ fixed) which magnifies the crossover region around the origin of the eigenvalue distribution. The proof exploits the fact that the three-term difference equation for orthogonal polynomials reduces into a universal second-order differential (Bessel) equation in the microscopic limit.

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The concept of universality in the theory of random matrices, or the independence of relevant quantities upon details of the potential function, is crucial in its application toward the level statistics of disordered physical systems. The universality of the macroscopic bulk two-point function in the large-$N$ limit is discovered to hold [1] as a direct consequence of the linear functional relationship (Cauchy inversion) between the potential $V(M)$ and the large-$N$ spectral density $\rho(z) = \lim_{N\to\infty} \langle 1/N \,\mathrm{tr}\, \delta(z-M) \rangle$

$$\frac{V'(z)}{2} = \fint_{-a}^{a} \frac{\mathrm{d}w}{z-w}\, \rho(w) \quad \Leftrightarrow \quad \rho(z) = -\frac{1}{\pi^2} \fint_{-a}^{a} \frac{\mathrm{d}w}{z-w} \sqrt{\frac{a^2-z^2}{a^2-w^2}} \frac{V'(w)}{2} \tag{1}$$

[2], and that $\rho(z,w) \sim \delta\rho(z)/\delta V(w)$. An alternative proof using orthogonal polynomials is found in ref.[3].

Another class of universality of a different origin, termed microscopic, has been anticipated whenever a quantity in concern is governed by the microscopic repulsion (anti-crossing) between energy levels dominantly enough to surpass the effect of a slowly varying potential. This type of (conjectured) universalities includes the appearance of the sine kernel for the

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microscopic correlators in the bulk [4,5]* (proved in ref.[6]), of the Airy kernel in the vicinity of a 'soft edge' ($\rho(z) \sim \sqrt{a-z}$) [7]* (proved in ref.[8]), and of the Bessel kernel of a 'hard edge' ($\rho(z) \sim 1/\sqrt{a-z}$) [9,10]*. The last and open problem has attracted considerable attention from condensed matter physics on spin-impurity scattering [11] as well as from high energy physics on QCD chiral symmetry breaking [12], and will be the central subject of this article.

The problem can equivalently be formulated in terms of chiral complex matrix models as follows: Consider a matrix integral with a generic potential

$$Z = \int \mathrm{d}^{2N^2} M \, \exp\left\{-\frac{N}{2} \mathrm{tr}\, V(M^2)\right\}, \quad V(M^2) = \sum_{k\geq 1} \frac{g_k}{k} M^{2k} \tag{2}$$

where $M$ stands for a $2N \times 2N$ block hermitian matrix whose non-zero components are $N \times N$ complex matrices on the off-diagonals,

$$M = \begin{pmatrix} 0 & C^\dagger \\ C & 0 \end{pmatrix}. \tag{3}$$

A 'chiral' complex matrix model (or chiral unitary ensemble) is so called because of the invariance under the transformation

$$C \mapsto U\, C\, V^\dagger, \quad U, V \in \mathrm{U}(N). \tag{4}$$

Since $M$ anticommutes with $\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, each of its eigenvalues $z$ always accompanies its mirror image $-z$ in the spectrum. The repulsion between these pairs is expected to yield a region in the vicinity of the origin where eigenvalues avoid populating. Magnify this region by measuring the correlation functions in the unit of an average level spacing $\sim 1/N$, i. e. by substituting $z = \zeta/N$. Are all the correlators independent of the potential in the limit $N \to \infty$, when $\zeta$ is kept fixed?

The answer has already been conjectured affirmative on several grounds [13,14]. First, one naïvely expects that, in the vicinity of the origin, the potential can essentially be regarded as a constant. Therefore it would not affect the microscopic correlation due to the level repulsion, except via the average level spacing used as a unit, which is determined by global balancing. Ref.[13] noticed that one and the same Bessel kernel emerges from two ensembles with distinct potentials, Gaussian $V = M^2$ and an infinite well*. This suggests that the Bessel kernel holds for a generic potential. Ref.[14] has calculated the one-point function for $V = M^2 + g\, M^4$ up to $O(g^1)$, which again supports the conjectured universality. We shall give a rigorous proof of this conjecture by exploiting the fact that the three-term

* These results for simple potentials are corollaries to the well-known asymptotic behaviour of classical orthogonal polynomials [15].

difference equations for orthogonal polynomials, characteristic of one-matrix models, universally reduces the Bessel equation in the above mentioned microscopic limit.

The partition function for a chiral complex matrix model (2) is expressible in terms of the component matrices as well as of the eigenvalues after integration over the angular coordinates $(U, V) \in \mathrm{U}(N) \times \mathrm{U}(N)/\mathrm{U}(1)^N$,

$$
\begin{aligned}
Z &= \int \mathrm{d}^{N^2} C^\dagger \ \mathrm{d}^{N^2} C \ \mathrm{e}^{-N \mathrm{tr} V(C^\dagger C)} \\
&= \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left( \mathrm{d}z_i^2 \ \mathrm{e}^{-NV(z_i^2)} \right) \prod_{i<j} \left| z_i^2 - z_j^2 \right|^2 = \int_0^{\infty} \prod_{i=1}^{N} \left( \mathrm{d}\lambda_i \ \mathrm{e}^{-NV(\lambda_i)} \right) \prod_{i<j} |\lambda_i - \lambda_j|^2 .
\end{aligned} \tag{5}
$$

The above expression can be interpreted as a *positive definite* hermitian matrix model in $H = C^\dagger C$ whose eigenvalues are $\lambda_1, \cdots, \lambda_N \geq 0$. In other words, the problem reduces to finding a set of orthogonal polynomials $P_n(\lambda)$ over the semi-infinite interval $[0, \infty)$,

$$
\int_0^{\infty} \mathrm{d}\lambda \ \mathrm{e}^{-NV(\lambda)} \, P_n(\lambda) \, P_m(\lambda) = h_n \, \delta_{nm}. \tag{6}
$$

We normalise them such that $P_n(0) = 1$ for later convenience,

$$
P_n(\lambda) = 1 + \cdots + p_n \lambda^n. \tag{7}
$$

Here it is assumed possible to choose this normalisation, which will prove equivalent to the ansatz that the origin be included in the support of the large-$N$ spectral density of $M$. Then the recursion relation for these $P_n$'s reads

$$
\begin{aligned}
\lambda \, P_n(\lambda) &= -q_n P_{n+1}(\lambda) + s_n P_n(\lambda) - q_{n-1} \frac{h_n}{h_{n-1}} P_{n-1}(\lambda) \qquad \left( q_n \equiv -\frac{p_n}{p_{n+1}} \right) \\
&\equiv \sum_m \hat{\lambda}_{nm} \, P_m(\lambda).
\end{aligned} \tag{8}
$$

The sets of unknowns $\{h_n\}$, $\{q_n\}$, $\{s_n\}$ are iteratively determined by [16]

$$
1 = -\int_0^{\infty} \mathrm{d}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \left\{ \mathrm{e}^{-N \, V(\lambda)} \, P_n(\lambda) \, P_n(\lambda) \right\} = N \, V'(\hat{\lambda})_{nn} \, h_n, \tag{9}
$$

$$
1 = -\int_0^{\infty} \mathrm{d}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \left\{ \mathrm{e}^{-N \, V(\lambda)} \, P_n(\lambda) \, P_{n-1}(\lambda) \right\} = \left( N \, V'(\hat{\lambda})_{nn-1} + \frac{n}{q_{n-1}} \right) h_{n-1} \tag{10}
$$

and eq.(8) at $\lambda = 0$ ,

$$0 = -q_n + s_n - q_{n-1} \frac{h_n}{h_{n-1}}. \tag{11}$$

We can immediately eliminate $s_n$'s using (11) to get

$$\lambda\, P_n(\lambda) = -q_n \left\{ P_{n+1}(\lambda) - P_n(\lambda) - \frac{h_n}{q_n} \frac{q_{n-1}}{h_{n-1}} \left( P_n(\lambda) - P_{n-1}(\lambda) \right) \right\}. \tag{12}$$

In the following we need to know the asymptotic behaviour of $q_n$ and $h_n$ for

$$n,\ N \to \infty \ \text{ while } \ \frac{n}{N} = t \text{ is kept fixed.} \tag{13}$$

Eqs.(9), (10) and (12) tell us that they should behave as[†]

$$q_n = q(\frac{n}{N}) + \text{higher orders in } \frac{1}{n}, \quad N\, h_n = h(\frac{n}{N}) + \text{higher orders in } \frac{1}{n}. \tag{14}$$

Then the matrix $\hat{\lambda}$ and its powers are approximated to be

$$\begin{aligned} \hat{\lambda}_{nm} &= q(\frac{n}{N}) \left( -\delta_{n\,m-1} + 2\delta_{nm} - \delta_{n\,m+1} \right), \\ (\hat{\lambda}^k)_{nm} &= q(\frac{n}{N})^k \sum_{\ell=-k}^{k} (-)^\ell \binom{2k}{k+\ell} \delta_{n\,m+\ell} \end{aligned} \tag{15}$$

so that eqs.(9) and (10) read

$$\sum_k g_k \binom{2k-2}{k-1} q(t)^{k-1} = \frac{1}{h(t)}, \tag{16}$$

$$-\sum_k g_k \binom{2k-2}{k} q(t)^{k-1} = \frac{1}{h(t)} - \frac{t}{q(t)}. \tag{17}$$

By eliminating $h(t)$ out of the above two, we obtain an algebraic equation for $q(t)$,

$$\frac{1}{2} \sum_k g_k \binom{2k}{k} q(t)^k = t. \tag{18}$$

Eqs.(16) and (18) imply a universal relationship among total derivatives,

$$\mathrm{d}t = 2q\ \mathrm{d}\left(\frac{1}{h}\right) + \frac{1}{h}\, \mathrm{d}q = 2\sqrt{q}\ \mathrm{d}\left(\frac{\sqrt{q}}{h}\right). \tag{19}$$

[†] $q_n$ and $h_n$ converge to smooth functions when the eigenvalues are supported on a single interval.

Next we expand the rhs of the recursion equation (12) in terms of $1/n$ in the limit (13),

$$\lambda\, P(n,N,\lambda) = -\frac{q(t)}{N^2}\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{h}{q}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{q}{h}\right)\frac{\mathrm{d}}{\mathrm{d}t}\right\} P(n,N,\lambda) + \text{higher orders in } \frac{1}{n} \tag{20}$$

where the argument $N$ in $P(n,N,\lambda) \equiv P_n(\lambda)$ is to indicate explicitly the dependency via the coefficient in front of the potential. It equivalently reads (subleading terms suppressed)

$$\left(h(t)\frac{\mathrm{d}}{\mathrm{d}t}\frac{q(t)}{h(t)}\frac{\mathrm{d}}{\mathrm{d}t} + N^2\lambda\right) P(n,N,\lambda) = 0 \tag{21}$$

telling us that that the arguments of $P$ appear only in the combinations $t = n/N$ and $x = N^2\lambda$ in the limit (13). The rescaled eigenvalue coordinate $x$ is to be fixed finite hereafter, and is regarded as a parameter in the ordinary differential equation in $t$. Performing the change of variable $t \mapsto u(t) \equiv \sqrt{q(t)/h(t)}$, using the relationship (19) and neglecting higher order terms in $1/n$, eq.(21) reduces to the Bessel equation of zeroth order,

$$\left(\frac{1}{u}\frac{\mathrm{d}}{\mathrm{d}u}u\frac{\mathrm{d}}{\mathrm{d}u} + 4x\right) P(u,x) = 0. \tag{22}$$

The general solution to it is a linear combination of Bessel and Neumann functions

$$P(u,x) = c(x)\, J_0\left(2u\sqrt{x}\right) + c'(x)\, Y_0\left(2u\sqrt{x}\right). \tag{23}$$

The integration constants (functions in $x$) are completely fixed by the boundary condition at $t = n/N = 0$ ($u(0) = 0$),

$$P(0,x) = P_0(\lambda) = 1, \tag{24}$$

to be $c(x) = 1,\ c'(x) = 0$. By substituting $t = 1$ into (23) and its $t$-derivative, we establish the following lemma for the asymptotic behaviour of generic orthogonal polynomials over the semi-infinite range $[0,\infty)$ which are normalisable to $P_n(0) = 1$:

$$\lim_{N\to\infty} P_N\left(\frac{x}{N^2}\right) = J_0\left(2u(1)\sqrt{x}\right), \tag{25}$$

$$\lim_{N\to\infty} N\left(P_N\left(\frac{x}{N^2}\right) - P_{N-1}\left(\frac{x}{N^2}\right)\right) = -\sqrt{\frac{x}{q(1)}}\, J_1\left(2u(1)\sqrt{x}\right). \tag{26}$$

The parameters $u(1)$ and $q(1)$, through which the dependence upon the potential enters the asymptotic form of the orthogonal polynomial, have the following simple meaning. Comparison of eqs.(16) and (18) at $t = 1$ with an explicit expression for the rhs of eq.(1)

$$\rho(z) = \frac{\sqrt{a^2 - z^2}}{2\pi}\sum_k g_k \sum_{n=0}^{k-1}\binom{2n}{n}\left(\frac{a^2}{4}\right)^n z^{2k-2n-2}, \tag{27}$$

$$\frac{1}{2}\sum_k g_k \binom{2k}{k} \left(\frac{a^2}{4}\right)^k = 1 \tag{28}$$

enables us to relate the parameters with the edge of the support of the large-$N$ spectral density $\rho(z)$ of the chiral complex matrix and its value at the origin as

$$a = 2\sqrt{q(1)}, \quad \rho(z=0) = \frac{u(1)}{\pi}. \tag{29}$$

It is easy to check that the critical condition $\rho(0) = 0$ for the 1-/2-cut transition [17] is equivalent to $P_N(0) = 0$ for the conventional, monically normalised polynomials $P_n(\lambda) = \lambda^n + \cdots$. Thus, under the assumption that the normalisation (7) is possible, the constants $h(1)$ and $q(1)$ are determined positive, due to the identification (29) valid for $\rho(z)$ supported on a single interval. The positivity of the norm $h$ is necessary for consistency, whereas $q > 0$ signifies that the coefficients in $P_n(\lambda)$ are alternating and eventually causes an oscillatory microscopic spectral density as it should.

Now we recall the expression for the integration kernel $K_N(\lambda, \mu)$ associated with the eigenvalue problem for the positive definite hermitian matrix $H = C^\dagger C$,

$$\begin{aligned} K_N(\lambda,\mu) &= \mathrm{e}^{-\frac{N}{2}(V(\lambda)+V(\mu))} \, \frac{1}{N}\sum_{i=0}^{N-1} \frac{P_i(\lambda)P_i(\mu)}{h_i} \\ &= \mathrm{e}^{-\frac{N}{2}(V(\lambda)+V(\mu))} \, \frac{-q_{N-1}}{N\, h_{N-1}} \frac{P_N(\lambda)P_{N-1}(\mu) - P_{N-1}(\lambda)P_N(\mu)}{\lambda - \mu}. \end{aligned} \tag{30}$$

Here use is made of the Christoffel-Darboux identity. Plugging in the lemmata (25) and (26), we obtain a universal form of the kernel (called the Bessel kernel) in the microscopic limit

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} K_N\left(\frac{x}{N^2}, \frac{y}{N^2}\right) = \\ &-2u(1)\frac{J_0(2u(1)\sqrt{x})\sqrt{y}J_1(2u(1)\sqrt{y}) - \sqrt{x}J_1(2u(1)\sqrt{x})J_0(2u(1)\sqrt{y})}{x-y}. \end{aligned} \tag{31}$$

The $s$-point correlation function $\sigma_N$ ($\rho_N$) of eigenvalues of $H$ ($M$) is represented in terms of the kernel as

$$\sigma_N(\lambda_1, \cdots, \lambda_s) = \left\langle \prod_{a=1}^{s} \frac{1}{N} \mathrm{tr}\, \delta(\lambda_a - H) \right\rangle = \det_{1\le a,b\le s} K_N(\lambda_a, \lambda_b) \tag{32}$$

$$\rho_N(z_1, \cdots, z_s) = \left\langle \prod_{a=1}^{s} \frac{1}{2N} \mathrm{tr}\, \delta(z_a - M) \right\rangle = |z_1| \cdots |z_s| \, \sigma_N(z_1^2, \cdots, z_s^2), \tag{33}$$

respectively. Therefore all the formulae for their microscopic limits

$$\varsigma(x_1,\cdots,x_s) \equiv \lim_{N\to\infty} \frac{1}{N^s}\sigma_N(\frac{x_1}{N^2},\cdots,\frac{x_s}{N^2}) \tag{34}$$

$$\varrho(\zeta_1,\cdots,\zeta_s) \equiv \lim_{N\to\infty} \rho_N(\frac{\zeta_1}{N},\cdots,\frac{\zeta_s}{N}) = |\zeta_1|\cdots|\zeta_s|\,\varsigma(\zeta_1^2,\cdots,\zeta_s^2), \tag{35}$$

previously calculated for the Laguerre (in the $H$-picture) or chiral Gaussian (in the $M$-picture) unitary ensemble, hold universally. Namely, the spectral density of the chiral complex matrix model

$$\rho_N(z) = \left\langle \frac{1}{2N}\operatorname{tr}\delta(z-M)\right\rangle = |z|K_N(z^2,z^2) \tag{36}$$

universally takes the form

$$\varrho(\zeta) = (\pi\rho(0))^2\,|\zeta|\left(J_0^2(2\pi\rho(0)\zeta) + J_1^2(2\pi\rho(0)\zeta)\right) \tag{37}$$

in the microscopic limit. It enjoys the matching condition between the micro- and macroscopic (large-$N$) spectral densities,

$$\lim_{\zeta\to\infty} \varrho(\zeta) = \rho(0). \tag{38}$$

In this article we have exhibited a proof of universality of the correlation functions of chiral complex matrix models in the microscopic limit. Consequently all dependencies of correlators upon the potential appears only through a single and local (at $z=0$) parameter $\rho(0)$ as anticipated. The universality holds as long as $\rho(0)>0$, i.e. the large-$N$ spectral density is supported on a single interval.

We have extended this strategy of taking the continuum limit of recursion equations, commonly used in the double-scaling calculations, to show microscopic universalities in a variety of matrix models which have been argued to be of relevance in QCD [18]. Details of these generalisations will appear elsewhere [19].

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